THE RETURN OF THE LINEAR SEARCH PROBLEM

BY

ANATOLE BECK AND PETER WARREN

ABSTRACT

The linear search problem concerns a search on the real line for a point selected at random according to a given probability distribution. The search begins at zero and is made by a continuous motion with constant speed, first in one direction and then the other. The problem is to determine when it is possible to devise a "best" search plan. In former papers the best plan has been selected according to the criterion of minimum expected path length. In this paper we consider a more general, nonlinear criterion for a "best" plan and show that the substantive requirements of the earlier results are not affected by these changes.

Introduction

In papers [1] and [2] the linear search problem is discussed and existence theorems for "best" search plans are established. The criterion used in both papers is that the expected path length required for finding the point is a minimum. If we imagine that searching becomes increasingly expensive in a nonlinear way with the length of the searching procedure, then this criterion is no longer valid. In this paper we consider a criterion which depends on the αth power of the path length, $\alpha \geqq 1$. We see that if we sharpen the analytical techniques of [1] and [2], then the results are unaltered in spite of the more stringent criterion.

Definitions and fundamental notions

Let $\alpha \geqq 1$ be fixed arbitrarily for the remainder of this paper. We will consider probability distributions F on the real line with a finite αth absolute moment $M_\alpha = M_\alpha(F) = \int_{-\infty}^{\infty} |t|^\alpha dF(T) < \infty$. If $0 \leqq \beta \leqq \alpha$, then $M_\beta < \infty$ since we always have $M_\beta \leqq 1 + M_\alpha$. F is assumed to be normalized, continuous from the left in the left half-line, continuous from the right in the right half-line, and continuous at 0, for reasons discussed in [1]. In that paper we define a *standard search procedure* as a sequence $x = \{x_i\}_{i=1}^{\infty}$ with

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$$\cdots \leqq x_4 \leqq x_2 \leqq 0 \leqq x_1 \leqq x_3 \leqq \cdots$$

or

$$\cdots \leqq x_3 \leqq x_1 \leqq 0 \leqq x_2 \leqq x_4 \leqq \cdots.$$

Standard search plans are called *strong* if all the inequalities are strict. Let $\mathfrak{X}_0$ designate the set of all standard search plans. In [2] we define a *generalized search procedure* as a sequence $x = \{x_i\}_{i=-\infty}^{+\infty}$ with

$$\cdots \leqq x_2 \leqq x_0 \leqq x_{-2} \leqq \cdots \leqq 0 \leqq \cdots \leqq x_{-1} \leqq x_1 \leqq x_3 \leqq \cdots.$$

Let $\mathfrak{X}_1$ designate the set of all generalized search plans. It is clear that we may embed $\mathfrak{X}_0$ in $\mathfrak{X}_1$ in a natural way by associating $x \in \mathfrak{X}_0$ with $\bar{x} \in \mathfrak{X}_1$ where $\bar{x}_0 = \bar{x}_{-1} = \bar{x}_{-2} = \cdots = 0$ and $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2$, etc.

Let t be a point between x_k and x_{k+2} (including x_{k+2} but not x_k). Then the *path length function* will be given by

$$X(x,t) = \begin{cases} \sum_{i=0}^{k+1} 2\left|x_i\right| + \left|t\right| & \text{if } x \in \mathfrak{X}_0, \\ \sum_{i=-\infty}^{k+1} 2\left|x_i\right| + \left|t\right| & \text{if } x \in \mathfrak{X}_1. \end{cases}$$

If t is not searched by a plan x (that is, if t does not lie between x_k and x_{k+2} for any k), then we define $X(x,t) = \infty$. This function describes the length of the search procedure x, where the direction is reversed at each x_i, up to the point t. The *search criterion* is given by

$$X_\alpha(x) = \int_{-\infty}^{\infty} (X(x,t))^\alpha dF(t).$$

Let

$$m_{0\alpha} = m_{0\alpha}(F) = \inf\{X_\alpha(x) \colon x \in \mathfrak{X}_0\}$$

$$m_{1\alpha} = m_{1\alpha}(F) = \inf\{X_\alpha(x) \colon x \in \mathfrak{X}_1\}.$$

It is remarked that $m_{0\alpha}$ and $m_{1\alpha}$ are finite whenever M_α is. This is because $m_{1\alpha} \leqq m_{0\alpha} \leqq 9^\alpha M_\alpha$ when $x = \{x_i\} = \{(-2)^i\}$ (cf. Beck [1]). Also, as in [1], we define $x^+ = x^+(F)$ and $x^- = x^-(F)$ so that $F(t) = 0$ if $t < x^-$, $F(t) = 1$ if $t > x^+$ and $0 < F(t) < 1$ if $x^- < t < x^+$. Finally

$$F^-(0) = \lim_{t \to 0^-} \sup \frac{F(t) - F(0)}{t}, \qquad F^+(0) = \lim_{t \to 0^+} \sup \frac{F(t) - F(0)}{t}.$$

Our first objective is to generalize the content of [1] in which are given necessary and sufficient conditions for the attainment of a minimal "cost" or "payoff" with a standard search procedure. The crucial tool in [1] is Lemma 3 which here is considerably sharpened as Lemma 6. Except where there are significant changes in technique, proofs are omitted. Instead, references are made to the appropriate parts of [1] or [2] where the necessary changes are easy exercises and left to the reader.

Our intention now is to prove

THEOREM 1. *Let F be a probability distribution on the real line with $M_\alpha(F) < \infty$. Then there is a standard search plan y with $X_\alpha(y) \leqq X_\alpha(x)$ for all search plans $x \in \mathfrak{X}_0$ if and only if at least one of $\bar{F}^+(0)$ and $\bar{F}^-(0)$ is finite.*

The procedure is basically a compactness argument. Consider a sequence of search plans $x^{(n)} = \{x_i^{(n)}\}$ with $X_\alpha(x^{(n)}) \to m_{0\alpha}$. Under the right conditions $y = \lim_{n\to\infty} x_i^{(n)}$ exists and we prove that $X_\alpha(y) = m_{0\alpha}$. We begin with three technical lemmas.

LEMMA 2. *If $u \geqq 0$ and $\eta \geqq 0$, then*

$$\alpha\eta u^{\alpha-1} \leqq (u+\eta)^\alpha - u^\alpha < \alpha\eta(u+\eta)^{\alpha-1}.$$

PROOF. Since the derivative of u^α is non-negative and increasing, the lemma is an immediate consequence of the mean-value theorem. Q.E.D.

LEMMA 3. *Let N be a fixed number, $N \geqq 1$. Then there exists $K_0 > 0$ such that*

$$\int_{|t|>K} (|t| - NK)^{\alpha-1} dF(t) > \frac{1}{8} M_{\alpha-1}$$

for $K \leqq K_0$.

PROOF. Choose K_2 such that

$$\int_{|t|>NK} |t|^{\alpha-1} dF(t) \geqq \frac{1}{2} M_{\alpha-1}$$

for $K \leqq K_2$. It follows from the Monotone Convergence theorem that there exists $K_1 \leqq K_2$ such that

$$\int_{|t|>NK} (|t| - NK)^{\alpha-1} dF(t) > \frac{1}{2}\int_{|t|>NK} |t|^{\alpha-1} dF(t)$$

for $K \leqq K_1$. Since

$$\int_{|t| \leqq NK} \left|(|t| - NK)^{\alpha-1}\right| dF(t) \to 0 \quad \text{as} \quad K \to 0$$

it is possible to choose $K_0 \leqq K_1 \leqq K_2$ such that

$$\int_{|t| \leqq NK} (|t| - NK)^{\alpha-1} \,|\, dF(t) \leqq \frac{1}{8} M_{\alpha-1}$$

for $K \leqq K_0$. In this case

$$\int_{|t|>K} (|t| - NK)^{\alpha-1} dF(t) > \frac{1}{4} M_{\alpha-1} - \int_{K<|t| \leqq NK} \left|(|t| - NK)^{\alpha-1}\right| dF(t)$$

$$\geqq \frac{1}{8} M_{\alpha-1}. \qquad \text{Q.E.D.}$$

LEMMA 4. *For $0 < \eta \leqq 1$ and E any measurable set,*

$$\int_E (X(x,t) + \eta)^\alpha - (X(x,t))^\alpha dF(t) \leqq \alpha 2^{\alpha-1} \eta \left(1 + \int_{-\infty}^{\infty} (X(x,t))^\alpha dF(t)\right).$$

PROOF. If $X(x,t) \geqq 1$, then

$$\begin{aligned}(X(x,t) + \eta)^\alpha &\leqq (X(x,t) + \eta X(x,t))^\alpha \\ &= (X(x,t))^\alpha (1 + \eta)^\alpha \\ &\leqq (X(x,t))^\alpha (1 + \alpha 2^{\alpha-1} \eta).\end{aligned}$$

If $X(x,t) < 1$, then

$$\begin{aligned}(X(x,t) + \eta)^\alpha &= \sum_{k=1}^{\infty} \binom{\alpha}{k} (X(x,t))^{\alpha-k} \eta^k \\ &\leqq (X(x,t))^\alpha + \sum_{k=1}^{\infty} \binom{\alpha}{k} \eta^k \\ &= (X(x,t))^\alpha + (1+\eta)^\alpha - 1 \\ &\leqq (X(x,t))^\alpha + \alpha 2^{\alpha-1} \eta.\end{aligned}$$

Now

$$\begin{aligned}&\int_E (X(x,t) + \eta)^\alpha - (X(x,t))^\alpha dF(t) \\ &\leqq \int_{X(x,t)<1} \alpha 2^{\alpha-1} \eta \, dF(t) + \int_{X(x,t) \geqq 1} \alpha 2^{\alpha-1} \eta (X(x,t))^\alpha dF(t) \\ &= \alpha 2^{\alpha-1} \eta \left(\int_{X(x,t)<1} dF(t) + \int_{X(x,t) \geqq 1} (X(x,t))^\alpha dF(t)\right) \\ &\leqq \alpha 2^{\alpha-1} \eta \left(1 + \int_{-\infty}^{\infty} (X(x,t))^\alpha dF(t)\right). \qquad \text{Q.E.D.}\end{aligned}$$

LEMMA 5. *If $x^- = -\infty$, $x^+ = +\infty$, then we can find a sequence $\{b_i\}$ such that $\forall i$, $|x_i| < b_i < \infty$ holds for every search plan $x \in \mathfrak{X}_0$ with $X_\alpha(x) < 2m_{0\alpha}$.*

PROOF. See [1, Lemma 2].

LEMMA 6. *If $F^-(0) < D < \infty$, then we can find a $K > 0$ such that for all sequences $x \in \mathfrak{X}_0$ with $x_2 < 0 < x_1$ and $x_3 - x_4 < K$, we can form a sequence y by removing x_1 and x_2 from x which has the property that $X_\alpha(y) \leqq X_\alpha(x)$.*

PROOF. Here $y_i = x_{i+2}$, $\forall i$. Let $a = 2(x_1 - x_2)$ and $b = 2(x_3 - x_1)$. Then

$$(X(x,t))^\alpha - (X(y,t))^\alpha = \begin{cases} 0 & \text{if } 0 \leqq t \leqq x_1, \\ (X(x,t))^\alpha - (X(x,t)+b)^\alpha & \text{if } x_2 \leqq t < 0, \\ (X(x,t))^\alpha - (X(x,t)-a)^\alpha & \text{if } t \notin [x_2, x_1]. \end{cases}$$

Thus

$$X_\alpha(x) - X_\alpha(y) = \int_{x_2}^{0} (X(x,t))^\alpha - (X(x,t)+b)^\alpha dF(t)$$
$$+ \int_{t \notin [x_2, x_1]} (X(x,t))^\alpha - (X(x,t)-a)^\alpha dF(t).$$

We shall show that if K is small enough, this difference is positive. Let $K > 0$ be chosen such that

1) $$K < \frac{1}{2}$$

2) $$K < \frac{M_{\alpha-1}}{2^{\alpha+2}D}$$

3) $$(F(t) - F(0))/t < D, \ \forall -K < t < 0$$

4) $$\int_{|t|>a} (|t| - a)^{\alpha-1} dF(t) \geqq \frac{1}{8} M_{\alpha-1}$$

5) $$F(K) - F(0) < \frac{1}{2}(1 - F(0)).$$

For $x_2 \leqq t < 0$, $X(x,t) \leqq 2x_1 - x_2 < a < 2K < 1$ and $b \leqq 2K < 1$. From Lemma 2, it follows that

$$(X(x,t))^\alpha - (X(x,t)+b)^\alpha \geqq -\alpha b (X(x,t)+b)^{\alpha-1} \geqq -\alpha K 2^\alpha.$$

Since $-x_2 < a/2$,

$$\int_{x_2}^{0} (X(x,t))^\alpha - (X(x,t)+b)^\alpha dF(t) \geqq -\alpha 2^{\alpha-1} K(F(0)-F(x_2))$$
$$\geqq -\alpha 2^{\alpha-1} D K a.$$

Also $t \notin [x_2, x_1]$ if and only if $X(x,t) > a$, and $X(x,t) \geqq |t|$, which together with Lemmas 2 and 3 imply†

$$\int_{t \notin [x_2,x_1]} (X(x,t))^\alpha - (X(x,t)-a)^\alpha dF(t)$$
$$\geqq \alpha a \int_{X(x,t)>a} (X(x,t)-a)^{\alpha-1} dF(t)$$
$$\geqq \frac{1}{8}\alpha a M_{\alpha-1}.$$

Finally

$$X_\alpha(x) - X_\alpha(y) \geqq a\alpha\left(\frac{1}{8}M_{\alpha-1} - 2^{\alpha-1} D K\right)$$
$$\geqq 0.$$

Q.E.D.

LEMMA 7. *If $F^-(0) < \infty$, $\varepsilon > 0$ and x is any strong search plan, then we can find a search plan $y \in \mathfrak{X}_0$ such that $X_\alpha(y) < X_\alpha(x) + \varepsilon$, $y_1 > 0$ and $y_3 - y_4 \geqq K$, where $K = K(F)$ is defined in the proof of Lemma 6.*

PROOF. Perhaps $x_1 \geqq 0$. If not, let $z = \{z_1\}$ be chosen with $0 < z_1 < x_2$ and $z_i = x_{i-1}$, $\forall i \geqq 2$. If z_1 is small enough, then Lemma 4 assures that $X_\alpha(z) < X_\alpha(x) + \varepsilon$. From here on the proof is identical to Lemma 4 in [1]. Q.E.D.

LEMMA 8. *If $X_\alpha(x) < 2m_{0\alpha}$, $x_1 > 0$, $x_3 - x_4 \geqq K$, $x \in \mathfrak{X}_0$, and $x^- < a < 0 < b < x^+$, then $x_i \in [a,b]$ for only n_0 values of i at most, where n_0 depends only on F, a, b and K.*

PROOF. See [1, Lemma 5].

THEOREM 9. *Let F be our given distribution. If $x^- = -\infty$, $x^+ = +\infty$, and $F^-(0) < \infty$, then there exists a search plan $y \in \mathfrak{X}_0$ with $X_\alpha(y) = m_{0\alpha}$.*

PROOF. The proof is very similar to that of Beck [1]. First note that for any weak search plan x, we can find a strong search plan z with $X_\alpha(z) \leqq X_\alpha(x)$. This is done exactly as in [1, Th. 6].

† If $x_2 = x^-$ and $x_1 = x^+$, then the two integrals in the computation would both be zero, and the last inequality would be false. Condition 5 on K assures that $x_1 < x^+$ since $x_1 \leqq x_3 < x^+$

Let $x^{(n)} = \{x_i^{(n)}\}$ be chosen for each n in such a way that $X_\alpha(x^{(n)}) \to m_{0\alpha}$. Let K be chosen as before and let a sequence $\{\varepsilon_n\}$ be chosen with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$. By Lemma 7 and the remark above, we can choose a strong search plan $z^{(n)}$ based on $x^{(n)}$ with $z_1^{(n)} > 0$, $z_3^{(n)} - z_4^{(n)} \geqq K$, and $X_\alpha(z^{(n)}) < X_\alpha(x^{(n)}) + \varepsilon_n$, $\forall n$. Then $X_\alpha(z^{(n)}) \to m_{0\alpha}$, and for each i, $\{z_i^{(n)}\}$ is a bounded sequence. Using the diagonal method, we can extract a subsequence $\{z^{(n_j)}\}$ of $\{z^{(n)}\}$ with $\{z_i^{(n_j)}\}$ convergent for each i as $j \to \infty$. Without any loss of generality we assume $X_\alpha(z^{(n)}) < 2m_{0\alpha}$, $\forall n$, and $\{z^{(n)}\} = \{z^{(n_j)}\}$. Let $y_i = \lim_n z_i^{(n)}$, $\forall i$, and $y = \{y_i\}$. Note that each search plan $z^{(n)}$ satisfies Lemma 8. Since $x^- = -\infty$ and $x^+ = +\infty$, $z_i^{(n)}$ lies in any fixed finite interval around zero for at most n_0 values of i for all n. Here n_0 does not depend on n. It follows that $|y_k| \to \infty$ as $k \to \infty$.

We wish to show that $X_\alpha(y) = m_{0\alpha}$. Unfortunately it need not be true that $(X(z^{(n)},t))^\alpha \to (X(y,t))^\alpha$ for every $t \in R$. We remedy this by introducing $w_i^{(n)}$ chosen so that it has the same sign as $z_i^{(n)}$ and so that $|w_i^{(n)}| = \max\{|z_i^{(n)}|, |y_i|\}$. Then $w_i^{(n)} \to y_i$ as $n \to \infty$, $\forall i$ and $|w_i^{(n)}| \geqq |y_i|$.

Choose any $k > 0$, $\delta > 0$, and let n_0 be chosen so that $|z_i^{(n)} - y_i| < \delta$, $\forall n > n_0$, $\forall i = 1, \cdots, 2k$. Then for every $y_{2k} < t < y_{2k-1}$ we have, from Lemma 2,

$$(X(w^{(n)},t))^\alpha \leqq (X(z^{(n)},t) + 2k \cdot 2\delta)^\alpha$$
$$\leqq \begin{cases} (X(z^{(n)},t)) + \alpha 2^{\alpha+1} k\delta \, (X(z^{(n)},t))^{\alpha-1} & \text{if } X(z^{(n)},t) \geqq 4k\delta, \\ (8k\delta)^\alpha & \text{otherwise.} \end{cases}$$

Thus

$$\int_{y_{2k}}^{y_{2k-1}} (X(w^{(n)},t))^\alpha dF(t) \leqq \int_{y_{2k}}^{y_{2k-1}} (X(z^{(n)},t))^\alpha dF(t)$$
$$+ \delta\alpha 2^{\alpha+1} k \int_{-\infty}^{\infty} (X(z^{(n)},t))^{\alpha-1} dF(t) + (8k\delta)^\alpha$$

where

$$\int_{-\infty}^{\infty} (X(z^{(n)},t))^{\alpha-1} dF(t) \leqq 1 + \int_{-\infty}^{\infty} (X(z^{(n)},t))^\alpha dF(t)$$
$$\leqq 1 + X_\alpha(x^{(n)}) + \varepsilon_n$$
$$\leqq 1 + 2m_{0\alpha} + \varepsilon_n$$

for n large enough, say $n > n_1$. On the other hand, as $n \to \infty$, we have

$$(X(w^{(n)},t)) \to (X(y,t))^\alpha \quad \text{for every} \quad t \in R.$$

Thus for n large enough, say $n > n_2$, we have

$$\left| \int_{y_{2k}}^{y_{2k-1}} (X(y,t))^\alpha \, dF(t) - \int_{y_{2k}}^{y_{2k-1}} (X(w^{(n)},t))^\alpha dF(t) \right| < \delta.$$

Hence, for $n > \max\{n_0, n_1, n_2\}$, we have

$$\begin{aligned} \int_{y_{2k}}^{y_{2k-1}} (X(y,t))^\alpha dF(t) &< \int_{y_{2k}}^{y_{2k-1}} (X(w^{(n)},t))^\alpha dF(t) + \delta \\ &\leqq \int_{y_{2k}}^{y_{2k-1}} (X(z^{(n)},t))^\alpha dF(t) + \delta(1 + \varepsilon_n + 2m_{0\alpha}) + (8k\delta)^\alpha + \delta. \end{aligned}$$

Since $X_\alpha(z^{(n)}) \to m_{0\alpha}$ and $\varepsilon_n \to 0$ as $n \to \infty$, we have for each $\delta > 0$,

$$\int_{y_{2k}}^{y_{2k-1}} (X(y,t))^\alpha dF(t) \leqq m_{0\alpha} + 2\delta(1 + m_{0\alpha}) + (8k\delta)^\alpha$$

so that

$$\int_{y_{2k}}^{y_{2k-1}} (X(y,t))^\alpha dF(t) \leqq m_{0\alpha}, \qquad \forall k > 0.$$

Since $|y_k| \to \infty$ as $k \to \infty$, we have $X_\alpha(y) \leqq m_{0\alpha}$. On the other hand, y is a search plan so that $X_\alpha(y) \geqq m_{0\alpha}$. Q.E.D.

COROLLARY 10. *If in Theorem 9, the hypothesis $F^-(0) < \infty$ is replaced by $F^+(0) < \infty$, then the same conclusion follows.*

PROOF. It is clear by symmetry.

THEOREM 11. *Let F be our given distribution. If $-\infty < x^- < 0$, $x^+ = +\infty$ and $F^-(0) < \infty$, then there is a $y \in \mathfrak{X}_0$ with $X_\alpha(y) = m_{0\alpha}$.*

PROOF. See [1, Th. 8].

COROLLARY 12. *If in Theorem 11, the hypothesis $F^-(0) < \infty$ is replaced by $F^+(0) < \infty$, then the same conclusion follows.*

PROOF. See [1, Corollary 9].

COROLLARY 13. *If in Theorem 11 or Corollary 12, the hypothesis $-\infty < x^- < 0$, $x^+ = +\infty$ is replaced by $x^- = -\infty$, $0 < x^+ < +\infty$, then the conclusions still hold.*

PROOF. It is clear by symmetry.

THEOREM 14. *Let F be our given distribution. If $-\infty < x^- < 0 < x^+ < +\infty$ and $F^-(0) < \infty$, then we can find a search plan $y \in \mathfrak{X}_0$ with $X_\alpha(y) = m_{0\alpha}$.*

PROOF. See [1, Th. 11].

COROLLARY 15. *If in Theorem 14, the assumption $\bar{F}^-(0) < \infty$ is replaced by $\bar{F}^+(0) < \infty$, then the same conclusion follows.*

PROOF. It is clear by symmetry.

THEOREM 16. *Let F be a probability distribution with $\int_{-\infty}^{\infty} |t|^\alpha dF(t) < \infty$. Suppose $\bar{F}^-(0) = \bar{F}^+(0) = \infty$. Let $x \in \mathfrak{X}_0$ be any search procedure with $X_\alpha(x) < 2m_{0\alpha}$. Then there exists a search procedure $y \in \mathfrak{X}_0$ such that $X_\alpha(y) < X_\alpha(x)$.*

PROOF. Since $\bar{F}^-(0) = \bar{F}^+(0) = \infty \neq 0, x^- < 0 < x^+$. Thus any search procedure has at least two entries. Assume $x_1 > 0$; the other case is dual. Choose any y_1 with $x_2 < y_1 < 0$, let $y_i = x_{i-1}$, $\forall i \geqq 2$, and define $\eta = 2\,|y_1|$. Then for $y_1 \leqq t < 0$, $X(y,t) = |t|$ and $X(x,t) = 2\,|x_1| + |t|$. Otherwise, $X(y,t) = X(x,t) + \eta$.

Observe that $|t|^\alpha - (2|x_1| + |t|)^\alpha \leqq -(2|x_1|)^\alpha$. This and Lemma 4 yield

$$\begin{aligned} X_\alpha(y) - X_\alpha(x) &= \int_{y_1}^{0} |t|^\alpha - (2|x_1| + |t|)^\alpha dF(t) \\ &\quad + \int_{t \notin [y_2, 0]} (X(x,t) + \eta)^\alpha - (X(x,t))^\alpha dF(t) \\ &\leqq (2|x_1|)^\alpha (F(y_1) - F(0)) \\ &\quad + \eta(\alpha 2^{\alpha-1}\left(1 + \int_{-\infty}^{\infty} (X(x,t))^\alpha dF(t)\right) \end{aligned}$$

provided that $|y_1| < \frac{1}{2}$. Indeed, if y_1 is chosen properly, this difference will be negative. In particular, let y_1 be chosen so that $x_2 < y_1 < 0$ and

$$\frac{F(y_1) - F(0)}{y_1} > \frac{3\alpha\, 2^{\alpha-1}\left(1 + \int_{-\infty}^{\infty} (X(x,t))^\alpha dF(t)\right)}{(2|x_1|)^\alpha}. \qquad \text{Q.E.D.}$$

PROOF OF THEOREM 1. It follows directly from Theorems 9, 11, 14 and 16, and Corollaries 10, 12, 13 and 15. Q.E.D.

Our second objective is to treat the content of [2] where it is shown that, for generalized search plans, there always exists a minimizing plan. In those cases where either $\bar{F}^-(0) < \infty$ or $\bar{F}^+(0) < \infty$, we will show that the minimizing generalized search plan is essentially the same as those obtained previously. Again, we follow closely the procedure in [2].

THEOREM 17. *Let F be a probability distribution on the real line with*

$M_\alpha(F) < \infty$. Then there is a generalized search plan y with $X_\alpha(y) \leqq X_\alpha(x)$ for all generalized search plans x.

LEMMA 18. *Assume $x^- = -\infty$, and $x^+ = +\infty$. Then for every $\varepsilon > 0$, there exists $B(\varepsilon) < \infty$ such that for all $x \in \mathfrak{X}_1$ with $X_\alpha(x) < 2\, m_{1\alpha}$, we have that $|x_j| < \varepsilon$ implies $|x_{j+1}| < B(\varepsilon)$.*

PROOF. See [2, Lemma 1].

LEMMA 19. *If $x^- < a \leqq b < x^+$, then there exists $B(a,b) < \infty$ such that for every $x \in \mathfrak{X}_1$ with $X_\alpha(x) < 2\, m_{1\alpha}$, we have that $x_j \in [a,b]$ implies $|x_{j+1}| < B(a,b)$.*

PROOF. See [2, Lemma 2].

COLRLLARY 20. *If $x^- = -\infty$, $x^+ = +\infty$, then for every $\varepsilon > 0$, there exists $C(\varepsilon) < \infty$ such that for every $x \in \mathfrak{X}_1$, with $X_\alpha(x) < 2\, m_{1\alpha}$, we have $|x_j| < \varepsilon$ implies $|x_{j+2}| < C(\varepsilon)$.*

PROOF. See [2, Corollary 3].

DEFINITION 21. Let ε_0 be any number such that $0 < \varepsilon_0 < x^+$, to be fixed for the remainder of this paper. For $x \in \mathfrak{X}_1$, let n_0 be chosen so that $x_{n_0+1} > \varepsilon_0$, $x_i \leqq \varepsilon_0$, $\forall i \leqq n_0$. Define a new generalized plan $\tilde{x}$ by $\tilde{x}_i = x_{i+n_0}$. Note $X\,(x) = X_\alpha(\tilde{x})$.

LEMMA 22. *Let $0 < \varepsilon < \varepsilon_0$. Then there is an $n = n(\varepsilon, F) > 0$ such that for every $x \in \mathfrak{X}_1$, with $X_\alpha(x) < 2\, m_{1\alpha}$, we have $|\tilde{x}_i| < \varepsilon$ for $\forall i < -n$.*

PROOF. See [2, Lemma 5].

LEMMA 23. *If $x^- < a \leqq 0 \leqq b < x^+$, then there exists $n = n(a,b,F) > 0$ such that for all $x \in \mathfrak{X}_1$ with $X_\alpha(x) < 2\, m_{1\alpha}$, we have $\tilde{x}_j \notin [a,b]$ for $\forall j > n$.*

PROOF. See [2, Lemma 6].

THEOREM 24. *If $x^- = -\infty$, $x^+ = +\infty$, then there is a $y \in \mathfrak{X}_1$ such that $X_\alpha(y) = m_{1\alpha}$.*

PROOF. Again the proof is similar to [2, Th. 7]. Let $x^{(n)} = \{x_i^{(n)}\}_{i=-\infty}^{+\infty}$ be chosen from $\mathfrak{X}_1$ for each n in such a way that $X_\alpha(x^{(n)}) \to m_{1\alpha}$. Note that $X_\alpha(\tilde{x}^{(n)}) \to m_{1\alpha}$ also, so that there will be no loss of generality if we assume $x^{(n)} = \tilde{x}^{(n)}$, $\forall n$.

Now

$$0 \leqq \ldots \leqq x_{-3}^{(n)} \leqq x_{-1}^{(n)} \leqq \varepsilon_0, \ \forall n,$$

$$0 \leqq \ldots \leqq |x_{-2}^{(n)}| \leqq |x_0^{(n)}| \leqq B(\varepsilon_0), \ \forall n, \text{ by Lemma 18, and}$$

$$|x_1^{(n)}| < B(B(\varepsilon_0)), \ |x_2^{(n)}| < B(B(B(\varepsilon_0))), \text{ etc. } \forall n$$

so that for each i, $\{x_i^{(n)}\}$ is a bounded sequence, and thus contains a convergent subsequence. By the diagonal process, we extract a subsequence $\{x^{(n_j)}\}_{j=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ such that $\{x_i^{(n_j)}\}_{j=1}^{\infty}$ converges for each i, and such that $X_\alpha(x^{(n_j)}) < 2\,m_{1\alpha}$ for $\forall j$. For ease of notation, since there is no loss of generality, we assume $\{x^{(n)}\}$ is actually the chosen subsequence. For each i, let $y_i = \lim_{n\to\infty} x_i^{(n)}$. Then $\cdots y_{-2} \leqq y_0 \leqq y_2 \leqq \cdots \leqq 0 \leqq \cdots \leqq y_{-1} \leqq y_1 \leqq \cdots$. Furthermore, from Lemma 23, for each $-\infty < a \leqq 0 \leqq b < +\infty$, we have $y_i \notin (a,b)$ for $i > n(a,b,F)$ so that $|y_i| \to \infty$ as $i \to +\infty$. Also, by Lemma 22, $|y_{-i}| \leqq \varepsilon$ if $i > n(\varepsilon,F)$, so that $y_i \to 0$ as $i \to -\infty$. Finally, if we set $P = 1 - F(\varepsilon_0)$, we have

$$X(x^{(n)},t) \geqq \sum_{i=-\infty}^{0} 2|x_i^{(n)}|, \qquad \forall t > \varepsilon_0, \qquad \forall n.$$

Thus

$$P\left(\sum_{i=-\infty}^{0} 2|x_i^{(n)}|\right)^\alpha < \int_0^{+\infty} (X(x^{(n)},t))^\alpha dF(t) < 2\,m_{1\alpha},$$

and

$$\sum_{i=-\infty}^{0} |x_i^{(n)}| < 2^{1/\alpha-1}\left(\frac{m_{1\alpha}}{P}\right)^{1/\alpha}, \qquad \forall n.$$

It follows that $\sum_{i=-\infty}^{0} |y_i| < \infty$ and $y \in \mathfrak{X}_1$. To show $X_\alpha(y) = m_{1\alpha}$, choose $\delta > 0$, and k large enough so that $\sum_{i=-\infty}^{-2k} |y_i| < \delta$. For each n, define $w^{(n)} \in \mathfrak{X}_1$ by

$$w_i^{(n)} = \begin{cases} x_i^{(n)} & \text{if } \ i < -2k \\ (-1)^{i+1}\max(|x_i^{(n)}|, |y_i|) & \text{if } \ -2k \leqq i \leqq 2k+1 \\ (-1)^{i+1}\max(|x_i^{(n)}|, |w_{i-2}|) & \text{if } \ i > 2k+1. \end{cases}$$

Then $w_i^{(n)} = x_i^{(n)}$ for all but at most s_k values of i, where $s_k = 4k + 1 + n(y_{2k}, y_{2k+1}, F)$. Choose any $\varepsilon > 0$ with $s_k\varepsilon < \delta$. Since $x_i^{(n)} \to y_i$, $\forall i$ and $x_i^{(n)} = w_i^{(n)}$ except for finitely many i, we know that for all n large enough, say $n > n_1$, we have $|w_i^{(n)} - x_i^{(n)}| < \varepsilon$, $\forall i$. Then from Lemma 2, we have

$$(X(w^{(n)},t))^\alpha < (X(x^{(n)},t) + s_k\varepsilon)^\alpha$$

$$\leqq \begin{cases} (X(x^{(n)},t))^\alpha + \alpha 2^{\alpha+1}(X(x^{(n)},t))^{\alpha-1}(s_k\varepsilon) & \text{if } X(x^{(n)},t) \geqq s_k\varepsilon, \\ (2s_k\varepsilon)^\alpha & \text{otherwise.} \end{cases}$$

Thus

$$X_\alpha(w^{(n)}) < X_\alpha(x^{(n)}) + \alpha 2^{\alpha+1}(s_k\varepsilon) \int_{-\infty}^{\infty} (X(x^{(n)},t))^{\alpha-1}\, dF(t) + (2s_k\varepsilon)^\alpha$$

where

$$\int_{-\infty}^{\infty} (X(x^{(n)},t))^{\alpha-1}\, dF(t) \leqq 1 + X_\alpha(x^{(n)}) < 1 + 2m_{1\alpha}.$$

Note that $w_i^{(n)} \to y_i$ as $n \to \infty$, $\forall i$, and define $v^{(n)} = \{v_i^{(n)}\}_{i=-\infty}^{\infty}$ in $\mathfrak{X}_1$ by

$$v_i^{(n)} = \begin{cases} y_i & \text{if } i < -2k, \\ w_i^{(n)} & \text{if } i \geqq -2k. \end{cases}$$

Then, for all t, we have from Lemma 2,

$$(X(v^{(n)},t))^\alpha \leqq (X(w^{(n)},t) + \sum_{i=-\infty}^{-2k} 2|y_i|)^\alpha$$

$$\leqq (X(w^{(n)},t) + 2\delta)^\alpha$$

$$\leqq (X(x^{(n)},t) + s_k\varepsilon + 2\delta)^\alpha$$

$$\leqq (X(x^{(n)},t) + 3\delta)^\alpha$$

$$\leqq \begin{cases} (X(x^{(n)},t))^\alpha + \alpha 2^{\alpha+1}(3\delta)\,(X(x^{(n)},t))^{\alpha-1} & \text{if } X(x^{(n)},t) \geqq 3\delta, \\ (6\delta)^\alpha & \text{otherwise.} \end{cases}$$

Hence

$$X_\alpha(v^{(n)}) \leqq X_\alpha(x^{(n)}) + \alpha 2^{\alpha+1}(3\delta) \int_{-\infty}^{\infty} (X(x^{(n)},t))^{\alpha-1}\, dF(t) + (6\delta)^\alpha$$

$$\leqq X_\alpha(x^{(n)}) + \alpha 2^{\alpha+1}(3\delta)(1 + 2m_{1\alpha}) + (6\delta)^\alpha.$$

Also, it is clear that $X(v^{(n)},t) \to X(y,t)$ uniformly for $y_{2k} \leqq t \leqq y_{2k+1}$. Thus

$$\int_{y_{2k}}^{y_{2k+1}} (X(y,t))\,dF(t)^{\alpha} = \lim_{n\to\infty} \int_{y_{2k}}^{y_{2k+1}} X(v^{(n)},t)\,dF(t)$$
$$\leqq \lim_{n\to\infty} \sup X_\alpha(v^{(n)})$$
$$\leqq \lim_{n\to\infty} \sup X_\alpha(x^{(n)}) + g(\delta)$$
$$< m_{1\alpha} + g(\delta)$$

where $g(\delta) = \alpha 2^{\alpha+1}(3\delta)(1+2m_{1\alpha}) + (6\delta)^{\alpha}$ and $g(\delta)$ decreases monotonically to zero as $\delta \to 0$. Since this inequality holds for all k large enough, and $|y_i| \to \infty$ as $i \to +\infty$, we have $X_\alpha(y) \leqq m_{1\alpha} + g(\delta)$. Since δ is arbitrary, we have $X_\alpha(y) \leqq m_{1\alpha}$ which gives us $X_\alpha(y) = m_{1\alpha}$. Q.E.D.

THEOREM 25. *If* $-\infty < x^- < 0$, $x^+ = +\infty$, *then there exists* $y \in \mathfrak{X}_1$ *such that* $X_\alpha(y) = m_{1\alpha}$.

PROOF. See [2, Th. 8].

COROLLARY 26. *If* $x^- = -\infty$, $0 < x^+ < +\infty$, *then there exists* $y \in \mathfrak{X}_1$ *such that* $X_\alpha(y) = m_{1\alpha}$.

PROOF. It follows by symmetry.

THEOREM 27. *If* $-\infty < x^- < 0 < x^+ < +\infty$, *then there exists a* $y \in \mathfrak{X}_1$ *such that* $X_\alpha(y) = m_{1\alpha}$.

PROOF. See [2, Th. 10].

PROOF OF THEOREM 17. It follows directly from Theorem 24, 25, 27 and Corollary 26.

In Theorem 1 we showed that under certain conditions on the distribution F, there is a $y \in \mathfrak{X}_0$ such that $X_\alpha(y) = m_{0\alpha}$. We conclude by considering the relationship between this result and Theorem 17.

LEMMA 28. $m_{0\alpha}(F) = m_{1\alpha}(F)$.

PROOF. See [2, Lemma 11] and use Lemma 2 of this paper. Q.E.D.

THEOREM 29. *Assume* $F^+(0) < \infty$. *Let* $y \in \mathfrak{X}_1$ *be such that* $X_\alpha(y) = m_{0\alpha} = m_{1\alpha}$. *Then there is a* k *with* $-\infty < k < +\infty$ *such that* $y_i = 0, \forall i \leqq k$.

PROOF. Assume not. Then $y_i \neq 0$, $\forall -\infty < i < +\infty$. Choose $D > 0$ with $F^+(0) < D < \infty$ and let $K > 0$ be chosen satisfying

i) $$\frac{F(t) - F(0)}{t} < D \text{ for } 0 < t < K,$$

ii) $$K < \frac{M_{\alpha-1}}{2^{\alpha+1}D},$$

iii) $$F(K)-F(-K) < \frac{1}{2},$$

v) $$\int_{t\notin[-K,K]} (|t|-4K)^{\alpha-1}dF(t) > \frac{1}{8}M_{\alpha-1},$$

where the possibility of the choice in (iv) is assured by Lemma 3.

Since $\sum_{i=-\infty}^{0}|y_i| < \infty$, we may choose an odd, negative number k such that

$$y_k - y_{k+1} = |y_k| + |y_{k+1}| < K \text{ and } \sum_{i=-\infty}^{k+1} |y_i| < \frac{1}{2}.$$

We shall show that $y_k = y_{k-1} = 0$. Define $x \in \mathfrak{X}_1$ by

$$x_i = \begin{cases} y_i, & \forall i > k, \\ y_{i-2}, & \forall i \leqq k. \end{cases}$$

Let $a = 2(y_k - y_{k-1})$, $b = 2(|y_{k+1}| - |y_{k-1}|)$. Then

$$(X(y,t))^\alpha - (X(x,t))^\alpha = \begin{cases} (X(y,t))^\alpha - (X(y,t)+b)^\alpha & \text{if } y_{k-2} \leqq t \leqq y_k, \\ (X(y,t))^\alpha - (X(y,t)-a)^\alpha & \text{if } t \notin [y_{k-1}, y_k], \\ 0 & \text{otherwise.} \end{cases}$$

Since $y_{k-1} \geqq -K$, $y_k \leqq K$, we have from Lemmas 2 and 3 that

$$\begin{aligned}\int_{t\notin[y_{k-1},y_k]} (X(y,t))^\alpha &- (X(y,t)-a)^\alpha dF(t) \\ &\geqq \alpha\, a \int_{t\notin[y_{k-1},y_k]} (X(y,t)-a)^{\alpha-1}\, dF(t) \\ &\geqq \alpha\, a \int_{t\notin[-K,K]} (X(y,t)-a)^{\alpha-1}\, dF(t) \\ &\geqq \alpha\, a \int_{t\notin[-K,K]} (|t|-4K)^{\alpha-1}\, dF(t) \\ &\geqq \frac{1}{8}\, \alpha\, a\, M_{\alpha-1}.\end{aligned}$$

Also by Lemma 2 and since, for $y_{k-2} \leqq t \leqq y_k$, $X(y,t) \leqq 2\sum_{i=-\infty}^{k+1} |y_i| < 1$ and $b < 1$, we have

$$\int_{y_{k-2}}^{y_k} (X(y,t))^\alpha - (X(y,t)+b)^\alpha dF(t) \geqq -\alpha b \int_{y_{k-2}}^{y_k} (X(y,t)+b)^{\alpha-1} dF(t)$$
$$\geqq -\alpha b\, 2^{\alpha-1}(F(y_k) - F(y_{k-2})).$$

Therefore

$$X_\alpha(y) - X_\alpha(x) \geqq \alpha\left[\tfrac{1}{4} M_{\alpha-1}(y_k - y_{k-1}) - 2^\alpha (y_{k+1} - y_{k-1})(F(y_k) - F(y_{k-2}))\right].$$

We observe the following:

a) $y_k - y_{k-1} \geqq y_k$ with equality only if $y_{k-1} = 0$.

b) $F(y_k) - F(y_{k-2}) \leqq F(y_k) - F(0) \leqq D\, y_k$ with equality only if $y_k = 0$.

c) $|y_{k+1}| - |y_{k-1}| < K$.

Now, by assumption on y,

$$0 \geqq X_\alpha(y) - X_\alpha(x)$$
$$\geqq \alpha(\tfrac{1}{4} M_{\alpha-1} y_k - 2^\alpha K D\, y_k)$$
$$\geqq 0$$

from (ii). It may be seen that equality cannot hold if either $y_k - y_{k-1} > y_k$ or $F(y_k) - F(y_{k-2}) < D\, y_k$. Thus equality holds only if $y_k = y_{k-1} = 0$, Q.E.D.

References

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University of Wisconsin
and
University of Denver